Integrable lattice equations and their growth properties

S. Tremblay
Centre de Recherches Mathématiques
et Département de Physique
Université de Montréal
C.P. 6128, Succ. Centre-ville
Montréal, H3C 3J7, Canada

B. Grammaticos
GMPIB, Université Paris VII
Tour 24-14, 5^{e} étage, case 7021
75251 Paris, France

A. Ramani
CPT, Ecole Polytechnique
CNRS, UMR 7644
91128 Palaiseau, France

Abstract

In this paper we investigate the integrability of two-dimensional partial difference equations using the newly developed techniques of study of the degree of the iterates. We show that while for generic, nonintegrable equations, the degree grows exponentially fast, for integrable lattice equations the degree growth is polynomial. The growth criterion is used in order to obtain the integrable deautonomisations of the equations examined. In the case of linearisable lattice equations we show that the degree growth is slower than in the case of equations integrable through Inverse Scattering Transform techniques.

The study of integrability of nonlinear evolution equations has spurred the development of efficient tools for its detection. The ARS [1] conjecture was formulated originally for partial differential equations and related integrability to the Painlevé property. In the discrete domain the singularity confinement [2] property was discovered while studying the lattice KdV equation and the singularities that can appear spontaneously during the evolution. The singularity confinement has been a most useful discrete integrability criterion in the sense that it is a necessary condition for lattice equations to be integrable by Inverse Scattering Transform (IST) methods. However, it has turned out [3] that singularity confinement is not sufficient for integrability and thus its use as an integrability detector must be subject to particular caution.

Another property of integrable discrete systems, namely the growth of the degree of the iterates [4], has, in the long run, proven to be a reliable integrability detector. The main idea goes back to Arnold [5] and Veselov [6]. As Veselov summarized it: "integrability has an essential correlation with the weak growth of certain characteristics". The characteristic quantity which can be easily obtained and computed for a rational mapping is the degree of the numerators or denominators of (the irreducible forms of) the iterates of some initial condition. (In order to obtain the degree one must introduce homogeneous coordinates and compute the homogeneity degree). Those ideas were refined by Viallet and collaborators [7,8], leading to the introduction of the notion of algebraic entropy. The latter is defined as $E = \lim_{n\to\infty} \log(d_n)/n$ where d_n is the degree of the n-th iterate. A generic, nonintegrable, mapping leads to exponential growth of the degrees of the iterates and thus has a nonzero algebraic entropy, while an integrable mapping has zero algebraic entropy. As we have shown in a previous work [9], this is too crude an estimate. The degree growth contains information that can be an indication as to the precise integration method to be used and thus should be studied in detail. (At this point, we must stress that, as was already pointed out in [8], the degrees of the iterates are *not* invariant under transformation of the variables. However the degree *growth* is invariant and characterises the system at hand).

In previous works of ours we have applied the techniques of degree growth to the study of one-dimensional mappings [9,10]. A first important conclusion of these studies was the confirmation of the singularity confinement results [11] on the derivation of discrete Painlevé equations. We have shown that, when singularity confinement is used for the deautonomisation of an integrable autonomous mapping, the condition obtained is identical to the one found by requiring nonexponential growth of the degrees of the iterates. (The terms "degrees of the iterates" in the above sentence and in the rest of the paper must be understood as the common homogeneity degree of the numerators and denominators of their irreducible forms, obtained through the introduction of the homogeneous coordinates). This not only confirms the results previously obtained for discrete Painlevé equations, but also suggests a

dual strategy for the study of discrete integrability based on the combined use of singularity confinement and study of degree growth. The second result [9] was that mappings which are linearisable are associated to a degree growth slower than the ones integrable through IST techniques. Thus, the detailed study of the degree is not only an indication of integrability but also of the integration method.

In this paper, we apply the techniques of degree growth to two-dimensional partial difference equations. We shall show that the main conclusions from the study of one-dimensional mappings carry over to the two-dimensional case in a rather straightforward way.

Let us start with the examination of the equation that serves as a paradigm in all integrability studies, namely KdV, the discrete form of which is [12,13]:

$$X_{n+1}^{m+1} = X_n^m + \frac{1}{X_{n+1}^m} - \frac{1}{X_n^{m+1}}. \tag{1}$$

(Incidentally, this is precisely the equation we have studied in [2], while investigating the singularity confinement property.) The study of the degree growth of the iterates in the case of a 2-dimensional lattice is substantially more difficult than that of the 1-dimensional case. It is thus very important to make the right choices from the outset. Here are the initial conditions we choose: on the line $m = 0$ we take X_n^0 of the form $X_n^0 = p_n/q$ while on the line $n = 0$ we choose $X_0^m = r_m/q$ (with $r_0 = p_0$). We assign to q and the p's, r's the same degree of homogeneity. Then we compute the iterates of X using (1) and calculate the degree of homogeneity in p, q, r at the various points of the lattice. Here is what we find:

$$\begin{array}{cccccccc}
 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \iddots \\
 & 1 & 7 & 19 & 31 & 41 & 51 & \cdots \\
 & 1 & 5 & 13 & 19 & 25 & 31 & \cdots \\
 & 1 & 3 & 5 & 7 & 9 & 11 & \cdots \\
m\uparrow & 1 & 1 & 1 & 1 & 1 & 1 & \cdots \\
 & \xrightarrow[n]{} & & & & & &
\end{array}$$

At this point we must indicate how the analytical expression for the degree can be obtained. First we compute several points on the lattice which allow us to have a good guess at how the degree behaves. In the particular case of a 2-dimensional discrete equation relating four points on an elementary square like (1), and with the present choice of initial conditions (and given our experience on 1-dimensional mappings) we can reasonably surmise that the dominant behaviour of the degree will be of the form $d_n^m \propto mn$. Moreover the subdominant

terms must be symmetric in m, n and at most linear. With those indications it is possible to "guess" the expression $d_n^m = 4mn - 2\max(m,n) + 1$ (for $mn \neq 0$) and subsequently calculate some more points in order to check its validity. This procedure will be used throughout this paper.

So the lattice KdV equation leads, quite expectedly, to a polynomial growth in the degrees of the iterates. Let us now turn to the more interesting question of deautonomisation. The form (1) of KdV is not very convenient and thus we shall study its potential form [14]:

$$x_{n+1}^{m+1} = x_n^m + \frac{z_n^m}{x_n^{m+1} - x_{n+1}^m}. \tag{2}$$

(The name 'potential' is given here in analogy to the continuous case: the dependent variable x of equation (2) is related to the dependent variable X of equation (1) through $x_n^{m+1} - x_{n+1}^m = X_n^m$ and (1) is recovered exactly if z_n^m=1). The deautonomisation we are referring to consists in finding an explicit m, n dependence of z_n^m which is compatible with integrability. Let us first compute the degrees of the iterates for constant z:

$$\begin{array}{c|ccccccc}
 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \iddots \\
 & 1 & 4 & 7 & 10 & 13 & 16 & \cdots \\
 & 1 & 3 & 5 & 7 & 9 & 11 & \cdots \\
 & 1 & 2 & 3 & 4 & 5 & 6 & \cdots \\
m\uparrow & 1 & 1 & 1 & 1 & 1 & 1 & \cdots \\
 & \xrightarrow[n]{} & & & & & &
\end{array}$$

The degree d_n^m is given simply by $d_n^m = mn + 1$. Assuming a generic (m, n) dependence for z we obtain the following successive degrees:

$$\begin{array}{c|ccccccc}
 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \iddots \\
 & 1 & 4 & 10 & 20 & 35 & 56 & \cdots \\
 & 1 & 3 & 6 & 10 & 15 & 21 & \cdots \\
 & 1 & 2 & 3 & 4 & 5 & 6 & \cdots \\
m\uparrow & 1 & 1 & 1 & 1 & 1 & 1 & \cdots \\
 & \xrightarrow[n]{} & & & & & &
\end{array}$$

We remark readily that the degrees form a Pascal triangle i.e. they are identical to the binomial coefficients, leading to an exponential growth at least on a strip along the diagonal.

The way to obtain an integrable deautonomisation is to require that the degrees obtained in the autonomous and nonautonomous cases be identical. The first constraint can be obtained by reducing the degree of x_2^2 from 6 to 5. As a matter of fact, starting from the initial conditions $x_n^0 = p_n/q$, $x_0^m = r_m/q$ (with $r_0 = p_0$) we obtain $x_1^1 = (p_1p_0 - p_0r_1 - z_0^0q^2)/(q(p_1 - r_1))$, $x_1^2 = Q_3/(qQ_2)$ where Q_k is a polynomial of degree k, and a similar expression for x_2^1. Computing x_2^2 we find $x_2^2 = Q_6/(q(p_1 - r_1)Q_4)$. It is impossible for q to divide Q_6 for generic initial conditions. However, requiring $(p_1 - r_1)$ to be a factor of Q_6 we find the constraint $z_1^1 - z_0^1 - z_1^0 + z_0^0 = 0$. The relation of this result to singularity confinement is quite easy to perceive. The singularity corresponding to $q = 0$ is indeed a fixed singularity: it exists for *all* (n, m)'s where either n or m are equal to zero. On the other hand the singularity related to $p_1 - r_1 = 0$ appears only at a certain iteration and is thus movable. The fact that with the proper choice of z_n^m the denominator factors out, is precisely what one expects for the singularity to be confined.

Requiring that z satisfy

$$z_{n+1}^{m+1} - z_n^{m+1} - z_{n+1}^m + z_n^m = 0 \tag{3}$$

suffices to reduce the degrees of all higher x's to those of the autonomous case. The solution of (3) is $z_n^m = f(n) + g(m)$ where f, g are two arbitrary functions. This form of z_n^m is precisely the one obtained in the analysis of convergence acceleration algorithms [15] using singularity confinement. The integrability of the nonautonomous form of (2) (and its relation to cylindrical KdV) has been discussed by Nagai and Satsuma [16] in the framework of the bilinear formalism.

We must point out here that the kind of initial conditions we choose, while influencing the specific degrees obtained, do not modify the conclusions on the type of growth. Let us illustrate this by choosing for (2) a staircase type of initial conditions where $x_n^{-n} = p_n/q$, $x_n^{1-n} = r_n/q$ with the same convention as to the degrees of q and the p's, r's (but without the now unnecessary constraint $p_0 = r_0$). We find the degrees:

$$\begin{array}{cccccccc}
 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \iddots \\
 & 1 & 2 & 4 & 7 & 11 & 16 & \cdots \\
 & 1 & 1 & 2 & 4 & 7 & 11 & \cdots \\
 & & 1 & 1 & 2 & 4 & 7 & \cdots \\
 & & & \underline{1} & 1 & 2 & 4 & \cdots \\
m\uparrow & & & & 1 & 1 & 2 & \cdots \\
 & \xrightarrow[n]{} & & & & & &
\end{array}$$

where the underlined 1 corresponds to the origin. The growth is again quadratic and depends only on the sum $N = n + m$ of the coordinates: $d_n^m = 1 + N(N-1)/2$.

Two more well-known discrete equations can be treated along the same lines. In the case of the lattice mKdV [14]:

$$x_{n+1}^{m+1} = x_n^m \, \frac{x_n^{m+1} - z_n^m x_{n+1}^m}{z_n^m x_n^{m+1} - x_{n+1}^m} \tag{4}$$

we obtain for constant z the same degree growth, $d_n^m = mn + 1$, as for the potential lattice KdV. If we assume now a generic z we find the degrees:

$$\begin{array}{ccccccccc}
 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \iddots \\
 & 1 & 4 & 13 & 32 & 65 & & \cdots \\
 & 1 & 3 & 7 & 13 & 21 & 31 & \cdots \\
 & 1 & 2 & 3 & 4 & 5 & 6 & \cdots \\
m\uparrow & 1 & 1 & 1 & 1 & 1 & 1 & \cdots \\
 & \xrightarrow[n]{} & & & & & &
\end{array}$$

The degrees obey the recursion $d_{n+1}^{m+1} = d_n^{m+1} + d_{n+1}^m + d_n^m - 1$ leading to an exponential growth with asymptotic ration $(1+\sqrt{2})$. Requiring the degree of x_2^2 to be 5 instead of 7 we find the condition

$$z_{n+1}^{m+1} z_n^m - z_n^{m+1} z_{n+1}^m = 0 \tag{5}$$

with solution $z_n^m = f(n)g(m)$. This condition is sufficient for the degrees of the nonautonomous case to coincide with those of the autonomous one. It is also precisely the one obtained in [15] using the singularity confinement condition. We believe that the Nagai-Satsuma approach [16] for the construction of double Casorati determinant solutions can be extended to the case of the nonautonomous lattice modified-KdV.

The discrete sine-Gordon equation [17,18]:

$$x_{n+1}^{m+1} x_n^m = \frac{1 + z_n^m x_n^{m+1} x_{n+1}^m}{x_n^{m+1} x_{n+1}^m + z_n^m} \tag{6}$$

in the autonomous case where z is a constant leads to the degree pattern:

$$\begin{array}{cccccccc}
 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \iddots \\
 & 1 & 5 & 9 & 13 & 16 & 19 & \cdots \\
 & 1 & 4 & 7 & 9 & 11 & 13 & \cdots \\
 & 1 & 3 & 4 & 5 & 6 & 7 & \cdots \\
m\uparrow & 1 & 1 & 1 & 1 & 1 & 1 & \cdots \\
 & \xrightarrow[n]{} & & & & & &
\end{array}$$

It can be represented by $d_n^m = mn + \min(m, n) + 1$. In the nonautonomous case of generic z we obtain the sequence of degrees:

$$\begin{array}{cccccccc}
 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \iddots \\
 & 1 & 5 & 19 & 49 & 96 & & \cdots \\
 & 1 & 4 & 11 & 19 & 29 & 41 & \cdots \\
 & 1 & 3 & 4 & 5 & 6 & 7 & \cdots \\
m\uparrow & 1 & 1 & 1 & 1 & 1 & 1 & \cdots \\
 & \xrightarrow[n]{} & & & & & &
\end{array}$$

obeying the relation $d_{n+1}^{m+1} = d_n^{m+1} + d_{n+1}^m + d_n^m - (1 - \delta_n^m)$ leading again to exponential growth. The condition for a growth identical to that of the autonomous case is the same as (5). Thus equation (6) introduces a nonautonomous extension of the lattice sine-Gordon equation. (We intend to return to a study of its properties in some future work). We must point out that in the continuous limit, this nonautonomous form goes over to $w_{x,t} = f(x)g(t)\sin w$. This explicit x and t dependence can be absorbed through a redefinition of the independent variables leading to the standard, autonomous, sine-Gordon, but no such gauge exists in the discrete case.

We now turn to two discrete equations which are particular in the sense that they are not integrable through IST techniques but rather through direct linearisation. The first is the discrete Liouville equation [19]:

$$x_{n+1}^{m+1} x_n^m = x_n^{m+1} x_{n+1}^m + z_n^m. \tag{7}$$

If we assume that z is a constant we obtain the following degree pattern:

$$\begin{array}{c|cccccccc}
 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \iddots \\
 & 1 & 4 & 5 & 6 & 7 & 8 & \cdots \\
 & 1 & 3 & 4 & 5 & 6 & 7 & \cdots \\
 & 1 & 2 & 3 & 4 & 5 & 6 & \cdots \\
m \uparrow & 1 & 1 & 1 & 1 & 1 & 1 & \cdots \\
 & \underset{n}{\longrightarrow} & & & & & &
\end{array}$$

By inspection we find $d_n^m = m + n$. This result is not at all astonishing. As we have shown in [9], the degree growth of linearisable mappings is slower than that of the IST integrable ones. The same feature appears again here. The deautonomisation of (7) can proceed along the same lines as previously. For generic z_n^m, the degrees are organised in a Pascal triangle and thus the growth is exponential. The condition for the growth to be identical to that of the autonomous case is again (5) and thus $z_n^m = f(n)g(m)$. However, this nonautonomous extension is trivial: it can be absorbed through a simple gauge transformation. Indeed, putting $x = \phi X$ where $\phi = \alpha(n)\beta(m)$ with $f(n) = \alpha(n)\alpha(n+1)$, $g(m) = \beta(m)\beta(m+1)$ we can reduce equation (7) to one where $z \equiv 1$.

Finally, we analyse the discrete Burgers equation [19]:

$$x_n^{m+1} = x_n^m \frac{1 + z_n^m x_{n+1}^m}{1 + z_n^m x_n^m}. \tag{8}$$

When z is a constant we find $d_n^m = m+1$. (Notice that contrary to all the previous examples, in the case of Burgers equation m and n do not play the same role and thus a d_n^m that is not symmetric in m, n is not surprising). For a generic z_n^m, we find $d_n^m = 2^m$, a manifestly exponential growth. The condition for the degree to grow like $m + 1$ is just

$$z_{n+1}^m - z_n^m = 0 \tag{9}$$

i.e. $z_n^m = g(m)$. This leads to a nonautonomous extension of the lattice Burgers equation. Moreover this extension cannot be removed by a gauge. On the other hand, this extension is perfectly compatible with linearisability. Indeed, putting $x_n^m = X_{n+1}^m / X_n^m$ we can reduce it to the linear equation:

$$X_n^{m+1} = f(m)(X_n^m + g(m) X_{n+1}^m) \tag{10}$$

where f is arbitrary and can be taken equal to unity. This nonautonomous extension is just a special case of the more general discrete Burgers:

$$x_n^{m+1} = x_n^m \frac{\alpha_n^m + \beta_n^m x_{n+1}^m}{1 + \gamma_n^m x_n^m} \tag{11}$$

which can be linearised through $x_n^m = \phi_n^m X_{n+1}^m / X_n^m$ to $X_n^{m+1} = \psi_n^m (X_n^m + \gamma_n^m \phi_n^m X_{n+1}^m)$ provided $\beta_n^m = \alpha_n^m \gamma_{n+1}^m$ and α, ϕ and ψ are related through $\alpha_n^m \psi_n^m \phi_n^m = \psi_{n+1}^m \phi_n^{m+1}$. We must point out here that the continuous Burgers equation also does possess a nonautonomous extension. It is straightforward to show that if ϕ is a solution of the equation $\phi_t = \phi^2 \phi_{xx}$ then the nonautonomous Burgers $u_t = \phi^2 u_{xx} + 2\phi u u_x$ can be linearised to $v_t = \phi^2 v_{xx}$ through the Cole-Hopf transformation $u = \phi v_x / v$.

In this paper, we have applied the method of the slow degree growth to the study of the integrability of partial difference equations. Our study has focused on well-known integrable lattice equations for which we have tried to provide nonautonomous forms. We have shown that using degree-growth methods it is possible to obtain integrable nonautonomous forms for most of the equations studied, and confirmed results previously obtained through the singularity confinement method. In the case of linearisable lattice equations, our results are the logical generalisation of the ones obtained for 1-dimensional mappings: the linearisable mappings have a degree growth that is slower than the one of the IST-integrable discrete equations. Our estimate of the degree growth was based on the direct computation of the degree for successive iterations and obtaining a fit of some analytical expression confirmed by subsequent iterations. It would be interesting, of course, to provide a rigorous proof of the degree growth following, for instance, the methods of [20]. However, this has not yet been carried through even for one-dimensional, nonautonomous mappings that are integrable through spectral methods. On the other hand, the proof of the degree growth for the cases where the equations are linearisable looks more tractable and we intend to address this question for both the one-and two-dimensional cases in some future work.

The fact that we were able, through the adequate choice of initial data, to perform these calculations without being overwhelmed by their size is an indication of the usefulness of our approach. The study of degree growth, perhaps coupled with singularity confinement in the dual strategy we sketched in [10], can be a precious tool for the detection of integrability of multidimensional discrete systems. The interest of this method is not only that it can be used as a detector of new integrable lattice systems but also that it can furnish an indication as to the precise method of their integration.

Acknowledgments

S. Tremblay acknowledges a scholarship from Centre de Cooperation Internationale Franco-Québecois.

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